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1994 J. Phys. A: Math. Gen. 27 2623

(http://iopscience.iop.org/0305-4470/27/8/003)

J. Phys. A: Math. Gen. 27 (1994) 2623–2631. Printed in the UK

Galilean limit of equilibrium relativistic mass distribution

L Burakovsky† and L P Horwitz‡

School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel-Aviv University, Tel-Aviv 69978, Israel

Received 16 November 1993, in final form 28 January 1994

Abstract. The low-temperature form of the equilibrium relativistic mass distribution is subject to the Galilean limit by taking $c \to \infty$. In this limit the relativistic Maxwell–Boltzmann distribution passes to the usual non-relativistic form and the Dulong–Petit law is recovered.

1. Introduction

In a previous paper [1] we studied an equilibrium relativistic ensemble, described by an equilibrium relativistic Maxwell–Boltzmann distribution with variable mass. For such a system a well defined mass distribution was found, consistent in low-temperature limit with the one obtained by Hakim [2] from the well known Jüttner–Synge distribution [3] of an on-mass-shell relativistic kinetic theory. Calculations of the average values of mass and energy gave in the low-temperature limit a correction of the order of 10% to the Dulong–Petit law.

In the present paper we consider the Galilean limit of the low-temperature form of the equilibrium relativistic mass distribution. We show that no correction to the Dulong–Petit law appears in this limit of the theory.

We remark that the low-temperature limit does not necessarily coincide with the non-relativistic limit of a theory. For example, very long wavelength radiation in Maxwell's relativistic theory does not coincide with the static Coulomb limit; it is still radiation. The limit must be carried out with special care to include a deformation of the Lorentz group to the Galilean group. The limit $c \to \infty$, carefully carried out, can do this, and this is the program of the present paper. The intrinsically relativistic mass distribution we found in the previous work, valid even at low temperatures, is deformed in the Galilean limit. The results we report here are known, of course, directly from the older Galilean statistical mechanics. Our purpose is to demonstrate the smoothness of this limit, and hence to show that the relativistic theory is a proper generalization of the idealized Galilean theory.

2. Preliminary remarks

In a previous paper, having begun with the low-temperature form of the relativistic Maxwell–Boltzmann distribution [1, equation (47)] (we use the metric $g^{\mu\nu} = (-,+,+,+)$, $q \equiv q^\mu$, $p \equiv p^\mu$, and take $\hbar = c = 1$ unless otherwise specified):

$$f(p,q) = C(q)\,\mathrm{e}^{-Am_c^2}\,\mathrm{e}^{2Ap_\mu p_c^\mu} \qquad (1)$$

† Bitnet: BURAKOV@TAUNIVM
‡ Bitnet: HORWITZ@TAUNIVM
Also at Department of Physics, Bar-Ilan University, Ramat-Gan, Israel.

which coincides with the Jüttner–Synge distribution adopted for an on-mass-shell relativistic kinetic theory. Here p^μ is the energy–momentum 4-vector of the one-body distribution, and $q \equiv q^\mu$ the spacetime position coordinates of the events of the ensemble. The constant entropy limit of the Maxwell–Boltzmann distribution implies that the distribution function contains a linear combination of the components of p^μ. The coefficient p_c^μ acquires interpretation, through the normalization of the distribution and Lorentz invariance, as proportional to the total energy–momentum of the system [1].

We obtained the following low-temperature form of the equilibrium relativistic mass distribution [1, equation (48)]:

$$f(m) = \frac{(2Am_c)^3}{2} m^2 K_1(2Am_c m) \tag{2}$$

where $m_c = \sqrt{-p_c^\mu p_{c\mu}}$ is a constant proportional to the total invariant mass of the N-body system and K_1 is a Bessel function of the third kind (imaginary argument),

$$K_\nu(z) = \frac{\pi \mathrm{i}}{2} \mathrm{e}^{\pi \mathrm{i}\nu/2} H_\nu^{(1)}(\mathrm{i}z).$$

In equation (2), $2Am_c$ corresponds to $1/k_\mathrm{B}T$ [1, equation (13)], i.e. $A = 1/2m_c k_\mathrm{B}T$ carries the temperature dependence of the distribution.

The distributions (1) and (2) give the following values of the average 4-momentum and mass:

$$\langle p^\mu \rangle = 4\frac{p_c^\mu}{m_c} k_\mathrm{B}T \qquad \langle m \rangle = \frac{3\pi}{4} k_\mathrm{B}T.$$

In the local rest frame we have

$$\langle E \rangle = 4k_\mathrm{B}T \tag{3}$$

and, consequently,

$$\langle E \rangle - \langle m \rangle = \gamma \tfrac{3}{2} k_\mathrm{B}T \tag{4}$$

where $\gamma = (16 - 3\pi)/6 \approx 1.1$ represents a relativistic correction to the Dulong–Petit law.

This result follows directly from equilibrium thermodynamics *without* imposing the geometrical restriction of the precise Galilean group to an infinitely sharp mass shell. As we have pointed out, the low-temperature limit of the relativistic theory does not automatically go to the Galilean limit (corresponding to a deformation of the symmetry group from Poincaré to Galilean invariance), i.e. the reason for the appearance of such a correction is determined by the fact that the difference $E - m$, even in the low-temperature limit, does not correspond to the expression for non-relativistic energy $\boldsymbol{p}^2/2m$. Therefore, the result that we have found cannot be considered as a non-relativistic limit; it is actually a relativistic low-temperature limit alone.

As is well known [4], the structure of the Galilean group, the symmetry of non-relativistic system, implies that the mass of a particle must be a constant intrinsic property.

In this paper we shall consider the Galilean limit of equilibrium relativistic ensemble which has been treated in the series of papers [1, 5], by taking $c \to \infty$ [6, 7]. We shall see that in the Galilean limit the difference $E - m$ approaches the non-relativistic expression $\boldsymbol{p}^2/2M$, where Galilean mass M coincides with the particle's intrinsic parameter. This variable $\boldsymbol{p}^2/2M$ turns out to be distributed over the ensemble with the usual non-relativistic Maxwell–Boltzmann distribution, due to the fact that the relativistic relation between the energy E and the mass m, $E^2 = m^2 + \boldsymbol{p}^2$ transforms in the Galilean limit to $E = m + \boldsymbol{p}^2/2M$, giving rise to the Maxwell–Boltzmann distribution of the latter. The first moment of this distribution $\langle p^2/2M\rangle$ (which coincides with $\langle E-m\rangle$), takes the value $\frac{3}{2}k_BT$, in agreement with the Dulong–Petit law.

We recognize, however, that the applicability of the Galilean group is an idealization of a world which seems to be more correctly described by the Poincaré group, and the Galilean limit is just a reasonable approximation for the relativistic relation $E^2 = m^2 + p^2$. The work reported here demonstrates that the relativistic theory is a proper generalization of the idealized Galilean theory studied in the non-relativistic framework, and approaches it smoothly as $c \to \infty$.

3. Galilean limit of a free relativistic N-particle system

We consider a system of N particles in the framework of a manifestly covariant mechanics [8], both for the classical theory and for the corresponding relativistic quantum theory. For the classical case, the dynamical evolution of such a system is governed by the equations of motion that are of the form of Hamilton equations for the motion of N *events* which generate the spacetime trajectories (particle world lines). These events are considered as the fundamental dynamical objects of the theory; they are characterized by their positions $q^\mu = (ct, \boldsymbol{r})$ and energy-momenta $p^\mu = (E/c, \boldsymbol{p})$ in an $8N$-dimensional phase space. The motion is parametrized by an invariant parameter τ [8], called the 'historical time'. The collection of events (called 'concatenation' [9]) along each world line corresponds to a *particle*, and hence the evolution of the state of the N-event system describes, *a posteriori*, the history in space and time of an N-particle system.

For the quantum case, the dynamical evolution is governed by a generalized Schrödinger equation for the wavefunction $\psi_\tau(q_1, q_2, \ldots, q_N) \in L^2(R^{4N})$, the Hilbert space of square integrable functions with measure $\mathrm{d}q_1\,\mathrm{d}q_2\ldots\mathrm{d}q_N \equiv \mathrm{d}^{4N}q$, describing the distribution of events and representing the probability amplitudes for finding events at spacetime points $(q_1^\mu, q_2^\mu, \ldots, q_N^\mu)$ at any instant τ:

$$i\hbar\frac{\partial\psi_\tau(q_1,\ldots,q_N)}{\partial\tau} = K\psi_\tau(q_1,\ldots,q_N)$$

where K is the dynamical evolution operator (generalized Hamiltonian), of the same form for both classical and quantum cases.

We shall consider here a many-particle system, within the framework of the relativistic generalization of the usual non-relativistic Boltzmann theory [5].

To study the non-relativistic limit of a dilute gas of events, it is sufficient to treat the simplest case of a system of N free particles with the Hamiltonian

$$K_0 = \sum_{i=1}^{N}\frac{p_{i\mu}p_i^\mu}{2M_i} \tag{5}$$

where M_i are positive parameters, the given intrinsic properties of the particles, having the dimension of mass.

The Hamilton equations

$$\frac{\mathrm{d}q_i^\mu}{\mathrm{d}\tau} = \frac{\partial K}{\partial p_{i\mu}} \qquad \frac{\mathrm{d}p_i^\mu}{\mathrm{d}\tau} = -\frac{\partial K}{\partial q_{i\mu}} \qquad i = 1, 2, \ldots, N$$

yield, in this case,

$$\frac{\mathrm{d}q_i^\mu}{\mathrm{d}\tau} = \frac{p_i^\mu}{M_i} \qquad \frac{\mathrm{d}p_i^\mu}{\mathrm{d}\tau} = 0 \qquad i = 1, 2, \ldots, N.$$

The evolution of the wavefunction is described by the equation

$$i\hbar\frac{\partial\psi_\tau(q_1,\ldots,q_N)}{\partial\tau}=K_0\psi_\tau(q_1,\ldots,q_N)\,. \qquad (6)$$

The wavefunction can be expressed as a Fourier transform

$$\begin{aligned}\psi_\tau(q_1,\ldots,q_N)&=\frac{1}{(2\pi\hbar)^{4N}}\int \mathrm{d}^4p_1\cdots\mathrm{d}^4p_N\,\mathrm{e}^{(\mathrm{i}/\hbar)\sum_{k=1}^N p_k^\mu q_{k\mu}}\psi_\tau(p_1,\ldots,p_N)\\&=\frac{1}{(2\pi\hbar)^{4N}}\int \mathrm{d}^4p_1\cdots\mathrm{d}^4p_N\,\mathrm{e}^{(\mathrm{i}/\hbar)\sum_{k=1}^N p_k^\mu q_{k\mu}}\mathrm{e}^{-(\mathrm{i}/\hbar)K_0\tau}\psi_0(p_1,\ldots,p_N)\\&=\frac{1}{(2\pi\hbar)^{4N}}\int \mathrm{d}^4p_1\cdots\mathrm{d}^4p_N\,\mathrm{e}^{(\mathrm{i}/\hbar)\sum_{k=1}^N (p_k^\mu q_{k\mu}-p_k^\mu p_{k\mu}/2M_k\tau)}\psi_0(p_1,\ldots,p_N)\,.\end{aligned} \qquad (7)$$

If this wavefunction is to be associated with particles, the function $\psi_\tau(p_1,\ldots,p_N)=\mathrm{e}^{-\mathrm{i}/\hbar K_0\tau}\psi_0(p_1,\ldots,p_N)$ must have support in momentum space in a region which is in the neighbourhood of definite masses (as pointed out in [7], these considerations are valid also in the presence of interaction, if it is not too strong). In the non-relativistic limit this support should approach the corresponding definite mass shells, consistent with a representation of the Galilean group.

We shall, therefore, require that the quantities

$$\epsilon_i=E_i-M_ic^2 \qquad i=1,\ldots,N \qquad (8)$$

constructed of variables occuring in the integrand of (7), i.e. in the support of $\psi_\tau(p_1,\ldots,p_N)$, be finite as $c\to\infty$ (compared to all other velocities) for the states with finite momenta.

We shall see that it is sufficient that the support of $\psi_\tau(p_1,\ldots,p_N)$ contract such that the variables

$$\eta_i=c^2(m_i-M_i) \qquad i=1,\ldots,N \qquad (9)$$

may take any value, however, finite, as $c\to\infty$; or, equivalently,

$$m_i=M_i\left(1+\mathrm{O}\left(\frac{1}{c^2}\right)\right). \qquad (10)$$

(The situation is quite similar to one in relativistic classical statistical mechanics, when this freedom permits one to obtain the Galilean microcanonical ensemble [6].)

Indeed, in this case one can show that the values $E_i-m_ic^2$ are equal to

$$E_i-m_ic^2=\frac{p_i^2}{2M_i}+\mathrm{O}\left(\frac{1}{c^2}\right) \qquad i=1,\ldots,N \qquad (11)$$

and approach non-relativistic kinetic energies of particles with the Galilean masses M_i as $c\to\infty$:

$$\begin{aligned}E_i-m_ic^2&=\sqrt{(m_ic^2)^2+p_i^2c^2}-m_ic^2\\&=\sqrt{(\eta_i+M_ic^2)^2+p_i^2c^2}-m_ic^2\\&=M_ic^2\sqrt{1+\frac{2\eta_i}{M_ic^2}+\frac{\eta_i^2}{M_i^2c^4}+\frac{p_i^2}{M_i^2c^2}}-m_ic^2\\&=M_ic^2\left(1+\frac{\eta_i}{M_ic^2}+\frac{p_i^2}{2M_i^2c^2}+\mathrm{O}\left(\frac{1}{c^4}\right)\right)-m_ic^2\end{aligned}$$

$$= (M_i c^2 + \eta_i) + \frac{p_i^2}{2M_i} - m_i c^2 + \mathrm{O}\left(\frac{1}{c^2}\right)$$
$$= \frac{p_i^2}{2M_i} + \mathrm{O}\left(\frac{1}{c^2}\right). \tag{12}$$

Consequently, the quantities

$$\epsilon_i = E_i - M_i c^2 = E_i - m_i c^2 + (m_i - M_i)c^2 = \frac{p_i^2}{2M_i} + \eta_i + \mathrm{O}\left(\frac{1}{c^2}\right) \qquad i = 1, \ldots, N \tag{13}$$

are finite as $c \to \infty$, as was required from the very beginning.

Now we turn to investigate the behaviour of the wavefunction $\psi_\tau(q_1, \ldots, q_N)$ in the Galilean limit:

$$\psi_\tau(q_1, \ldots, q_N) = \frac{1}{(2\pi\hbar)^{4N}} \int \mathrm{d}^4 p_1 \cdots \mathrm{d}^4 p_N \, \mathrm{e}^{(\mathrm{i}/\hbar)\sum_{k=1}^N (p_k r_k - E_k t_k)}$$
$$\times \mathrm{e}^{(\mathrm{i}/\hbar)\sum_{k=1}^N m_k^2 c^2/2M_k \tau} \psi_0(p_1, \ldots, p_N)$$
$$= \frac{1}{(2\pi\hbar)^{4N}} \int \frac{\mathrm{d}E_1}{c} \cdots \frac{\mathrm{d}E_N}{c} \mathrm{d}^3 p_1 \cdots \mathrm{d}^3 p_N \, \mathrm{e}^{(\mathrm{i}/\hbar)\sum_{k=1}^N [p_k r_k - (p_k^2/2M_k + M_k c^2 + \eta_k) t_k]}$$
$$\times \mathrm{e}^{(\mathrm{i}/\hbar)\sum_{k=1}^N (M_k c^2/2 + \eta_k)\tau} \psi_0(p_1, \ldots, p_N)$$
$$= \frac{1}{(2\pi\hbar)^{4N}} \int \frac{\mathrm{d}\eta_1}{c} \cdots \frac{\mathrm{d}\eta_N}{c} \mathrm{d}^3 p_1 \cdots \mathrm{d}^3 p_N \, \mathrm{e}^{(\mathrm{i}/\hbar)\sum_{k=1}^N (p_k r_k - p_k^2/2M_k t_k)}$$
$$\times \mathrm{e}^{-(\mathrm{i}/\hbar)\sum_{k=1}^N M_k c^2 t_k} \mathrm{e}^{(\mathrm{i}/\hbar)\sum_{k=1}^N (M_k c^2/2)\tau} \, \mathrm{e}^{(\mathrm{i}/\hbar)\sum_{k=1}^N (\tau - t_k)\eta_k} \psi_0(p_1, \ldots, p_N). \tag{14}$$

Although the support of $\psi_0(p_1, \ldots, p_N)$ is bounded in the η_k's as $c \to \infty$, the integrals over the η_ks can approximately yield factors of $\delta(t_k - \tau)$, as remarked in [7]. Consider the case for which $\psi_0(p_1, \ldots, p_N)$ is independent of η_k, for $k = 1, \ldots, N$, in $-_k \leqslant \eta_k \leqslant \Delta_k$ and is zero outside this region; then the wavefunction (14) is proportional to the product

$$\prod_{k=1}^N \int_{-\Delta}^{\Delta} \mathrm{d}\eta_k \, \mathrm{e}^{(\mathrm{i}/\hbar)\eta_k(\tau - t_k)} = \prod_{k=1}^N \int_{-\Delta/\hbar}^{\Delta/\hbar} \hbar \, \mathrm{d}\eta_k' \, \mathrm{e}^{\mathrm{i}\eta_k'(\tau - t_k)}$$
$$= (2\pi\hbar)^N \prod_{k=1}^N \delta_{\Delta/\hbar}(\tau - t_k) \tag{15}$$

where

$$\Delta = \min(\Delta_1, \Delta_2, \ldots, \Delta_N)$$

and

$$\delta_{\Delta/\hbar}(\tau - t_k) \to \delta(\tau - t_k)$$

if† $\hbar \to 0$ (it is clear now that $\Delta \to 0$ precisely would be an unsuitable condition for the non-relativistic limit). The dispersion of t_k around τ, bounded by

$$|t_k - \tau| \leqslant \hbar/\Delta_k \leqslant \hbar/\Delta$$

is therefore a purely quantum effect (it does not depend on c and vanishes with $\hbar \to 0$), emerging asymptotically from a relativistic quantum theory in the Galilean limit, as emphasised in [7].

† $\Delta \to \infty$ may also satisfy this condition; in the present paper we shall use the fact that Δ can take any infinitesimal value but not zero.

Thus we will have

$$\int \frac{\mathrm{d}\eta_1}{c} \cdots \frac{\mathrm{d}\eta_N}{c}\, \mathrm{e}^{(\mathrm{i}/\hbar)\sum_{k=1}^{N}(\tau - t_k)\eta_k} \cong (2\pi\hbar)^N \prod_{k=1}^{N} \delta(c\tau - ct_k) \tag{16}$$

which means that the times associated with all of the particles become synchronized in the Galilean limit:

$$ct_1 = ct_2 = \cdots = ct_N = ct = c\tau\,. \tag{17}$$

This result also can be obtained from the canonical equations of motion.

We have with the help of (8)

$$K_0 = \sum_{k=1}^{N} \frac{-E_k^2/c^2 + p_k^2}{2M_k} = \sum_{k=1}^{N} \left\{ -\frac{1}{2M_k c^2}\epsilon_k(\epsilon_k + 2M_k c^2) - \frac{M_k c^2}{2} + \frac{p_k^2}{2M_k} \right\}$$
$$= \sum_{k=1}^{N} \frac{p_k^2}{2M_k} - \sum_{k=1}^{N} \epsilon_k - \frac{c^2}{2}\sum_{k=1}^{N} M_k - \sum_{k=1}^{N} \frac{\epsilon_k^2}{2M_k c^2}\,. \tag{18}$$

Since, according to the equations of motion,

$$\frac{\mathrm{d}}{\mathrm{d}\tau} ct_k = -\frac{\partial K_0}{\partial (E_k/c)} = -\frac{\partial K_0}{\partial(\epsilon_k/c)} = c + \frac{\epsilon_k}{M_k c}$$

it follows that

$$ct_k = c\tau + \int_0^{\tau} \frac{\epsilon_k(\tau')}{M_k c}\,\mathrm{d}\tau' + ct_k(0)\,.$$

Now choosing in the initial instant $t_k(0) = t(0)$ for all of the particles, $k = 1, \ldots, N$, and taking $c \to \infty$, we obtain (17).

Finally, taking into account all of the above mentioned considerations, we can see that the initial wavefunction $\psi_\tau(q_1, \ldots, q_N)$ in the Galilean limit approaches the non-relativistic expression

$$\psi_\tau(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N, t) = \left(\frac{1}{2\pi\hbar}\right)^{3N}$$
$$\times \int \mathrm{d}^3\boldsymbol{p}_1 \cdots \mathrm{d}^3\boldsymbol{p}_N\, \mathrm{e}^{(\mathrm{i}/\hbar)\sum_{k=1}^{N}(\boldsymbol{p}_k\boldsymbol{r}_k - \boldsymbol{p}_k^2/2M_k t)}\, \mathrm{e}^{-(\mathrm{i}/\hbar)\varphi}\, \psi_0(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N) \tag{19}$$

up to an additional phase factor $\mathrm{e}^{-(\mathrm{i}/\hbar)\varphi}$, where

$$\varphi = Mc^2 t/2 \qquad M = \sum_{k=1}^{N} M_k\,.$$

4. Galilean limit of the relativistic Maxwell–Boltzmann distribution

We now wish to consider the Galilean limit of the low-temperature form of equilibrium relativistic mass distribution (2), which, including explicit factors of c, is

$$f(mc^2) = \frac{1}{2(k_\mathrm{B}T)^3}(mc^2)^2 K_1\left(\frac{mc^2}{k_\mathrm{B}T}\right)\,.$$

In the Galilean limit $c \to \infty$ the argument of the K-function goes to infinity. Using the asymptotic formula [10]

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z}\left\{1 + \frac{4\nu^2 - 1}{8z} + \cdots\right\} \qquad z \to \infty$$

we obtain

$$f(mc^2) \sim \frac{(mc^2)^{3/2}}{(k_B T)^{5/2}} \, e^{-mc^2/k_B T}$$

and after normalization we have the following mass distribution (we again suppress c):

$$f(m) = \frac{1}{\Gamma(\frac{5}{2})(k_B T)^{5/2}} m^{3/2} \, e^{-m/k_B T} . \tag{20}$$

This formula coincides with the corresponding limit of the mass distribution of Hakim [2]

$$f(m) = \frac{2}{3\pi (k_B T)^3} m^2 K_2 \left(\frac{m}{k_B T} \right)$$

(the limit $z \to \infty$ of $K_\nu(z)$ is independent of ν in leading order).

The distribution (20) gives the following average values:

$$\langle m \rangle = \tfrac{5}{2} k_B T \qquad \langle m^2 \rangle = \tfrac{35}{4} (k_B T)^2 \tag{21}$$

more generally,

$$\langle m^\ell \rangle = \frac{\Gamma(\ell + \frac{5}{2})}{\Gamma(\frac{5}{2})} (k_B T)^\ell . \tag{22}$$

Following the method of Hakim [2], we compute $\langle E \rangle$ with the distribution†

$$f(\varepsilon) = \left\langle \delta \left(\varepsilon + \frac{p^\mu p_{c\mu}}{m_c} \right) \right\rangle$$

i.e.

$$f(\varepsilon) \sim \int d^4 p \, \delta \left(\varepsilon + \frac{p^\mu p_{c\mu}}{m_c} \right) e^{2A p^\mu p_{c\mu}} \sim \varepsilon^3 \, e^{-\varepsilon / k_B T}$$

where the estimate is made for large c. This result provides the normalized distribution

$$f(E) = \frac{1}{3!(k_B T)^4} E^3 \, e^{-E/k_B T} . \tag{23}$$

It then follows that

$$\langle E^\ell \rangle = \frac{(\ell + 3)!}{3!} (k_B T)^\ell \tag{24}$$

and

$$\langle E \rangle = 4 k_B T \tag{25}$$

which, in fact, coincides in form with the low-temperature limit of the relativistic theory (3).

Therefore, taking into account (21) and (25), we obtain

$$\langle E \rangle - \langle m \rangle = \tfrac{3}{2} k_B T \tag{26}$$

in agreement with the Dulong–Petit law.

In conclusion we shall show that in the Galilean limit the variable $E - m = p^2/2M$ has the usual non-relativistic Maxwell–Boltzmann distribution.

In the framework of the relativistic Boltzmann theory [1, 5] particles are considered as having equal intrinsic parameters:

$$M_1 = M_2 = \cdots = M_N = M .$$

† The quantity ε coincides with E in the local rest frame.

It then follows from the relations

$$\eta = m - M \qquad -\Delta \leqslant \eta \leqslant \Delta$$

that

$$M - \Delta \leqslant m \leqslant M + \Delta . \tag{27}$$

From the normalization conditions for the relativistic Maxwell–Boltzmann distribution in the low-temperature limit [1, equations (2) and (47)], we have

$$n(q) = C(q)\,\mathrm{e}^{-Am_c^2} \int \mathrm{d}^4 p\, \mathrm{e}^{2Ap^{\mu}p_{c\mu}}$$

($n(q)$ is the total number of events per unit spacetime volume in the system in the neighbourhood of the point q). This integral written in the local rest frame $p_c^{\mu} = (m_c, 0)$ takes on the form ($2Am_c = 1/k_{\mathrm{B}}T$)

$$\int \mathrm{d}E\, \mathrm{d}^3 p\, \mathrm{e}^{-E/k_{\mathrm{B}}T} .$$

Taking into account (11) ad (27), one can rewrite this expression as follows:

$$\int \mathrm{d}^3 p\, \mathrm{d}m\, \mathrm{e}^{-(m+p^2/2M)/k_{\mathrm{B}}T} = \int \mathrm{d}^3 p\, \mathrm{e}^{-(p^2/2M)} \int_{M-\Delta}^{M+\Delta} \mathrm{d}m\, \mathrm{e}^{-m/k_{\mathrm{B}}T} .$$

The latter integral

$$\int_{M-\Delta}^{M+\Delta} \mathrm{d}m\, \mathrm{e}^{-m/k_{\mathrm{B}}T} = 2k_{\mathrm{B}}T\, \mathrm{e}^{-M/k_{\mathrm{B}}T} \sinh \frac{\Delta}{k_{\mathrm{B}}T}$$

does not vanish since Δ is finite (it enters the normalization factor). We see that the freedom of Δ to take any value (in this case not necessary infinitesimal but $\leqslant M$), finite as $c \to \infty$ but not equal to zero, enables one to obtain the non-relativistic Maxwell–Boltzmann distribution for $e \equiv p^2/2M$.

Finally, we have

$$n(q) = 2C(q)k_{\mathrm{B}}T \sinh \frac{\Delta}{k_{\mathrm{B}}T}\, \mathrm{e}^{-m_c/2k_{\mathrm{B}}T}\, \mathrm{e}^{-M/k_{\mathrm{B}}T} \int \mathrm{d}^3 p\, \mathrm{e}^{-p^2/2M} \tag{28}$$

which is the usual (normalized) non-relativistic Maxwell–Boltzmann distribution

$$f(e) = \frac{1}{\Gamma(\frac{3}{2})(k_{\mathrm{B}}T)^{3/2}}\, e^{1/2} \exp\left(-\frac{e}{k_{\mathrm{B}}T}\right).$$

5. Concluding remarks

We have considered the Galilean limit of equilibrium relativistic ensemble. We have found that the relativistic relation between the energy E and the mass m transforms in this limit to $E = m + p^2/2M$, giving rise to the non-relativistic Maxwell–Boltzmann distribution of $p^2/2M$. The first moment of this distribution $\langle p^2/2M \rangle$ (which coincides with $\langle E - m \rangle$) is equal to $\frac{3}{2}k_{\mathrm{B}}T$, in agreement with the Dulong–Petit law, and no relativistic correction appears in this limit, in contrast to [1].

For the case of an equilibrium relativistic ensemble of *indistinguishable events* [11] the distribution function is found to be

$$f(q, p) = C(q) \frac{1}{\mathrm{e}^{-A(p+p_c)^2} \pm 1} .$$

In the Galilean limit it becomes the usual non-relativistic distribution of Bose–Einstein or Fermi–Dirac, with chemical potential $\mu_G = \mu - (M/N)$ [12], where μ is the chemical potential of relativistic theory [6], M the Galilean mass and N is the total number of particles.

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